

Non-uniform Hyperbolicity and Non-uniform Specification

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Abstract

In this paper we deal with an invariant ergodic hyperbolic measure μ for a diffeomorphism f , assuming that f is either $C^{1+\alpha}$ or f is C^1 and the Oseledec splitting of μ is dominated. We show that this system (f, μ) satisfies a weaker and non-uniform version of specification, related with notions studied in several recent papers, including [20, 27, 19, 23, 25, 15].

Our main results have several consequences: as corollaries, we are able to improve the results about quantitative Poincaré recurrence, removing the assumption of the non-uniform specification property in the main Theorem of [20] that establishes an inequality between Lyapunov exponents and local recurrence properties. Another consequence is the fact that any of such measure is the weak limit of averages of Dirac measures at periodic points, as in [21]. Following [27] and [19], one can show that the topological pressure can be calculated by considering the convenient weighted sums on periodic points, whenever the dynamics is positive expansive and every measure with pressure close to the topological pressure is hyperbolic.

1 Introduction

In seminal works, Bowen [2, 3] and Sigmund [21] introduced in the 1970's the notion of *specification* and used this to show many ergodic properties for dynamical systems satisfying this property, including subshifts of finite type and sofic subshifts, the restriction of an axiom A diffeomorphism to its non wandering set, expanding differentiable maps, and geodesic flows on manifolds with negative curvature. Before continue, let us recall the definition of the *uniform* specification property (SP) as defined by Bowen in [2].

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We say that $f : X \rightarrow X$ has the *specification property* (SP) if given $\theta > 0$ there exists $K_\theta \in \mathbb{N}$ such that for all $x \in X$, there exists $p \in \mathbb{N}$ such that the dynamical ball

$$B_m^n(x, \theta) := \bigcap_{k=-m}^n f^{-k}(B(f^k(x), \theta))$$

contains a periodic point with period $p \leq n + m + K_\theta$.

While the specification property usually holds in symbolic dynamics, often in a trivial way, it is a strong hypothesis for dynamical systems. A very interesting result shows that for continuous $f : [0, 1] \rightarrow [0, 1]$, specification holds if and only if f is topologically mixing (see [1]). On the other hand, it is known that for β -shifts, specification occurs rarely: it is verified only on a set of β of Lebesgue measure zero (see [5]).

For maps that do not have specification, several weaker versions were introduced in the literature along the last decade. From the best of our knowledge, the first weaker version was introduced by Marcus at [14], to show that periodic measures are weakly dense for every ergodic toral automorphism, extending a previous work of Sigmund that proved the same for hyperbolic toral automorphisms. For the non-uniformly hyperbolic case, several versions of non-uniform specification, including [7, 11, 12, 22], were introduced and had been used to show many ergodic properties.

Saussol, Troubetzkoy and Vienti in [20] also introduced another non-uniform specification property and showed that every hyperbolic ergodic measure having non-uniform specification property satisfies an inequality between Lyapunov exponents and local recurrence rates. This notion is defined using *slow varying* functions that appear in the Pesin's Theory, i.e., functions q that satisfy $q(f^{\pm 1}(x)) \leq e^\eta q(x)$ for every $x \in M$. Let us define the non-uniform specification precisely.

Definition 1.1. *Let μ be an invariant measure of f , we say that (f, μ) satisfies non-uniform specification property (NS), if for μ almost every x , any small $\eta > 0$, any η -slowly varying positive function q , any integers m, n , and any $\theta > 0$ there exists $K := K(\eta, \theta, x, m, n)$ such that:*

(i) *the non-uniform dynamical ball*

$$\tilde{B}_m^n(x, \theta) := \bigcap_{k=-m}^n f^{-k} B(f^k(x), \theta q(f^k(x))^{-2})$$

contains a periodic point with period $p \leq n + m + K$;

(ii) *the dependence of K on m, n satisfies*

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} = 0.$$

In words: pieces of orbits with length n are shadowed by periodic orbits whose period is less than n plus a sub linear term with respect to n .

A natural question arises: whether this non-uniform specification property is valid for non-uniformly hyperbolic systems?

In [15], the first author had proved that every expanding ergodic measure for maps with non-flat critical set, i.e., a strongly mixing measure with positive exponents, has this non-uniform specification property. In this paper, we give a positive answer for hyperbolic ergodic measures to show that every hyperbolic ergodic measure naturally has the non-uniform specification property of [20] and thus we can remove the assumption of non-uniform specification property in the main theorem of [20] to establish an inequality between Lyapunov exponents and local recurrence properties and some other corollaries. Now we state our main results.

Theorem 1.2. *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism. Then, every f ergodic hyperbolic measure μ satisfies NS.*

Remark 1.3. In particular, if μ is a mixing hyperbolic measure, then we have a stronger version of the non-uniform specification property: for any $p \geq n + m + K$, the non-uniform dynamical ball $\tilde{B}_m^n(x, \theta)$ contains a periodic point with period p .

Remark 1.4. Note that if the function $q(\cdot) \equiv 1$, then the non-uniform dynamical ball $\tilde{B}_m^n(x, \theta, q)$ is the general well-known dynamical ball $B_m^n(x, \theta)$. So, in the conclusion of non-uniform specification property we can replace non-uniform dynamical balls by dynamical balls and η can be omitted.

Given $x \in M$ and $r > 0$, denote the first return time of a ball $B(x, r)$ radius r at x by

$$\tau(B(x, r)) := \min\{k > 0 \mid f^k(B(x, r)) \cap B(x, r) \neq \emptyset\}.$$

Then we can obtain a corollary as follows by using Theorem 1.2 and the main Theorem in [20].

Corollary 1.5. *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism. Then for every f ergodic hyperbolic measure μ , one has for μ a.e. $x \in M$,*

$$\limsup_{r \rightarrow 0} \frac{\tau(B(x, r))}{-\log r} \leq \frac{1}{\lambda_u} - \frac{1}{\lambda_s},$$

where λ_u, λ_s are the minimal positive Lyapunov exponent and maximal negative Lyapunov exponent of μ , respectively.

Remark 1.6. From the main Theorem in [20], if $h_\mu(f) > 0$, then for μ a.e. $x \in M$,

$$\liminf_{r \rightarrow 0} \frac{\tau(B(x, r))}{-\log r} \geq \frac{1}{\Lambda_u} - \frac{1}{\Lambda_s},$$

where Λ_u, Λ_s are the maximal positive Lyapunov exponent and minimal negative Lyapunov exponent of μ , respectively. Therefore, using our Corollary 1.5 we can get that if M is two dimensional and $h_\mu(f) > 0$, then for μ a.e. $x \in M$,

$$\lim_{r \rightarrow 0} \frac{\tau(B(x, r))}{-\log r} = \frac{1}{\lambda_u} - \frac{1}{\lambda_s} = \frac{1}{\Lambda_u} - \frac{1}{\Lambda_s}.$$

We observe that the specification property that we just discussed before has many similar different forms, as it is presented by [27]. Let us discuss a slightly generalization of the non-uniform specification above, that we call generalized non-uniform specification property with respect to several orbit segments (being a generalization of NS introduced in Definition 1.1).

Definition 1.7. (*Generalized Non-uniform Specification GNS*) We say that μ has the generalized non-uniform specification property if for μ almost every x , any small $\eta > 0$, any η -slowing varying function q (that is, $q(f^{\pm 1}(x)) \leq e^\eta q(x)$), any integer m, n , and any $\theta > 0$ there exists $K := K(\eta, \theta, x, m, n)$ satisfying

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} = 0$$

and so that the following holds: given points x_1, x_2, \dots, x_k in a full μ -measure set and positive integers $m_1, \dots, m_k, n_1, \dots, n_k$, there is $p_i \geq 0$ with

$$\sum_{i=1}^k p_i \leq \sum_{i=1}^k K(\eta, \theta, x_i, m_i, n_i)$$

(in particular if μ is mixing, for any $t \geq \sum_{i=1}^k K(\eta, \theta, x_i, m_i, n_i)$ there is p_i with $\sum_{i=1}^k p_i = t$) and a periodic point z with period $p = \sum_{j=1}^k (n_j + m_j + p_j)$ such that

$$z \in \tilde{B}_{m_1}^{n_1}(x_1, \theta) := \cap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta q(f^j(x_1))^{-2})$$

and for $2 \leq i \leq k$,

$$f^{\sum_{j=1}^{i-1} (n_j + p_j) + \sum_{j=2}^i m_j}(z) \in \tilde{B}_{m_i}^{n_i}(x_i, \theta) := \cap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta q(f^j(x_i))^{-2}).$$

One natural question that arises from Theorem 1.2 above is

Question 1.8. Let f be a C^1 diffeomorphism. Every f ergodic hyperbolic measure μ satisfies GNS?

In particular, if $q(x) \equiv 1$, the required result in Question 1.8 is obviously valid for uniformly hyperbolic systems [21], since it can be deduced from classical (uniform) specification property [21] for dynamical balls. Moreover, $K(\theta, x, m, n)$ can be chosen only dependent on θ from [21].

Here we show that the answer for the Question 1.8 is positive, either if f is $C^{1+\alpha}$ or if f is C^1 with dominated Oseledec splitting.

Theorem 1.9. Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism. Then every f ergodic hyperbolic measure μ satisfies GNS.

Remark 1.10. In particular, if $q(x) \equiv 1$, we also have a generalization of Remark 1.4 for the general well-known dynamical ball $B_m^n(x, \theta)$. I.e., in the conclusion of GNS we can replace non-uniform dynamical balls by dynamical balls and η can be omitted.

At the end of this section, we point out that the above results are also valid for C^1 non-uniformly hyperbolic systems with dominated splitting which is based on a recent result of [24]. Before that we recall the notion of dominated splitting. Let Δ be an f -invariant set and $T_\Delta M = E \oplus F$ be a Df -invariant splitting on Δ . $T_\Delta M = E \oplus F$ is called (S_0, λ) -dominated on Δ (or simply dominated), if there exist two constants $S_0 \in \mathbb{Z}^+$ and $\lambda > 0$ such that

$$\frac{1}{S} \log \frac{\|Df^S|_{E(x)}\|}{m(Df^S|_{F(x)})} \leq -2\lambda, \quad \forall x \in \Delta, \quad S \geq S_0.$$

Theorem 1.11. *Let f be a C^1 diffeomorphism. Then every ergodic hyperbolic measure μ in whose Oseledec splitting the stable bundle dominates the unstable bundle on $\text{supp}(\mu)$ satisfies GNS.*

2 Pesin theory

In this section we give a quick review concerning some notions and results of $C^{1+\alpha}$ Pesin theory. We point the reader to [9, 10, 18] for more details. Let $f : M \rightarrow M$ be a $C^{1+\alpha}$ diffeomorphism. We recall the concept of Pesin set and recall Katok's shadowing lemma in this section.

2.1 Pesin set

Given $\lambda, \mu \gg \varepsilon > 0$, and for all $k \in \mathbb{Z}^+$, we define $\Lambda_k = \Lambda_k(\lambda, \mu; \varepsilon)$ to be all points $x \in M$ for which there is a splitting $T_x M = E_x^s \oplus E_x^u$ with invariant property $D_x f^m(E_x^s) = E_{f^m x}^s$ and $D_x f^m(E_x^u) = E_{f^m x}^u$ satisfying:

- (a) $\|Df^n|_{E_{f^m x}^s}\| \leq e^{\varepsilon k} e^{-(\lambda-\varepsilon)n} e^{\varepsilon|m|}, \quad \forall m \in \mathbb{Z}, \quad n \geq 1;$
- (b) $\|Df^{-n}|_{E_{f^m x}^u}\| \leq e^{\varepsilon k} e^{-(\mu-\varepsilon)n} e^{\varepsilon|m|}, \quad \forall m \in \mathbb{Z}, \quad n \geq 1;$
- (c) $\tan(\angle(E_{f^m x}^s, E_{f^m x}^u)) \geq e^{-\varepsilon k} e^{-\varepsilon|m|}, \quad \forall m \in \mathbb{Z}.$

We set $\Lambda = \Lambda(\lambda, \mu; \varepsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k$ and call Λ a Pesin set.

According to Oseledec Theorem[17], every ergodic hyperbolic measure μ has s ($s \leq d = \dim M$) nonzero Lyapunov exponents

$$\lambda_1 < \cdots < \lambda_r < 0 < \lambda_{r+1} < \cdots < \lambda_s$$

with associated Oseledec splitting

$$T_x M = E_x^1 \oplus \cdots \oplus E_x^s, \quad x \in O(\mu),$$

where we recall that $O(\mu)$ denotes an Oseledec basin of μ . If we denote by λ the absolute value of the largest negative Lyapunov exponent λ_r and μ the smallest positive Lyapunov exponent λ_{r+1} , then for any $0 < \varepsilon < \min\{\lambda, \mu\}$, one has μ full-measure Pesin set $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$ (see, for example, Proposition 4.2 in [18]). And for any point $x \in O(\mu) \cap \Lambda$, E_x^u and E_x^s coincide with $E_x^1 \oplus \cdots \oplus E_x^r$ and $E_x^{r+1} \oplus \cdots \oplus E_x^s$ respectively.

The following statements are elementary properties of Pesin blocks (see [18]):

- (a) $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \subseteq \cdots$;
- (b) $f(\Lambda_k) \subseteq \Lambda_{k+1}$, $f^{-1}(\Lambda_k) \subseteq \Lambda_{k+1}$;
- (c) Λ_k is compact for $\forall k \geq 1$;
- (d) for $\forall k \geq 1$ the splitting $x \rightarrow E_x^u \oplus E_x^s$ depends continuously on $x \in \Lambda_k$.

2.2 Shadowing lemma

We recall Katok's shadowing lemma [18] in this subsection. Let $(\delta_k)_{k=1}^{+\infty}$ be a sequence of positive real numbers. Let $(x_n)_{n=-\infty}^{+\infty}$ be a sequence of points in $\Lambda = \Lambda(\lambda, \mu, \varepsilon)$ for which there exists a sequence $(s_n)_{n=-\infty}^{+\infty}$ of positive integers satisfying:

- (a) $x_n \in \Lambda_{s_n}$, $\forall n \in \mathbb{Z}$;
- (b) $|s_n - s_{n-1}| \leq 1$, $\forall n \in \mathbb{Z}$;
- (c) $d(fx_n, x_{n+1}) \leq \delta_{s_n}$, $\forall n \in \mathbb{Z}$;

then we call $(x_n)_{n=-\infty}^{+\infty}$ a $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit. Given $\theta > 0$, a point $x \in M$ is an τ -shadowing point for the $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit if $d(f^n x, x_{n+1}) \leq \tau \varepsilon_{s_n}$, $\forall n \in \mathbb{Z}$, where $\varepsilon_k = \varepsilon_0 e^{-\varepsilon k}$ and ε_0 is a constant only dependent on the system of f .

Lemma 2.1. (*Shadowing lemma*) *Let $f : M \rightarrow M$ be a $C^{1+\alpha}$ diffeomorphism, with a non-empty Pesin set $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$ and fixed parameters, $\lambda, \mu \gg \varepsilon > 0$. For $\forall \tau > 0$ there exists a sequence $(\delta_k)_{k=1}^{+\infty}$ such that for any $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit there exists a unique τ -shadowing point.*

3 Recurrence Times

In this section we always assume that $f : X \rightarrow X$ is a homeomorphism on a compact metric space, μ is an invariant measure and Γ is a subset of X with positive measure for μ . For $x \in \Gamma$, define

$$\cdots < t_{-2}(x) < t_{-1}(x) < t_0(x) = 0 < t_1(x) < t_2(x) < \cdots$$

to be the all times such that $f^{t_i}(x) \in \Gamma$ (called recurrence times). By Poincaré Recurrence Theorem, the sequence above is well-defined at μ a.e. $x \in \Gamma$. Note that

$$t_1(f^{t_i}(x)) = t_{i+1}(x) - t_i(x).$$

In general, we call t_1 to be the first recurrence time. An increasing sequence of natural numbers $\{N_i\}_{i \geq 1}$ is called *nonlacunary*, if $\lim_{i \rightarrow +\infty} \frac{N_{i+1}}{N_i} = 1$. For recurrence times, we have a basic proposition of their nonlacunary as follows.

Proposition 3.1. *For μ a.e. $x \in \Gamma$,*

$$\lim_{i \rightarrow +\infty} \frac{t_{i+1}(x)}{t_i(x)} = 1 \quad \text{and} \quad \lim_{i \rightarrow -\infty} \frac{t_{i-1}(x)}{t_i(x)} = 1.$$

In particular,

$$\lim_{i,j \rightarrow +\infty} \frac{t_{i+1}(x) - t_{j-1}(x)}{t_i(x) - t_j(x)} = 1.$$

Proof It can be proved by using Borel-Cantelli lemma and Kač Lemma(See [16], Proposition 3.8). Here we give another direct proof of the first equality which only depends on Birkhoff Ergodic Theorem. The proof of the remain equalities are similar.

By Birkhoff Ergodic Theorem, for any subset $A \subseteq M$, the limit function

$$\chi_A^*(x) := \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_A(f^j(x))$$

exists for μ a.e. x and f -invariant. Moreover, $\int \chi_A^*(x) d\mu = \int \chi_A(x) d\mu = \mu(A)$.

If $A := \Gamma$, we claim that for μ a.e. $x \in \Gamma$, $\chi_\Gamma^*(x) > 0$ (this is a basic fact of recurrence and we will give the proof below). Then, by the definition of $t_i(x)$, for μ a.e. $x \in \Gamma$,

$$\lim_{i \rightarrow +\infty} \frac{i}{t_i(x)} = \lim_{t_i(x) \rightarrow +\infty} \frac{1}{t_i(x)} \sum_{j=0}^{t_i(x)-1} \chi_\Gamma(f^j(x)) = \chi_\Gamma^*(x) > 0.$$

Thus, for μ a.e. $x \in \Gamma$,

$$\lim_{i \rightarrow +\infty} \frac{t_{i+1}(x)}{t_i(x)} = \lim_{i \rightarrow +\infty} \frac{t_{i+1}(x)}{i+1} \cdot \frac{i+1}{i} \cdot \frac{i}{t_i(x)} = \frac{1}{\chi_\Gamma^*(x)} \cdot 1 \cdot \chi_\Gamma^*(x) = 1.$$

Now we start to prove the claim. If μ is ergodic, it is obvious since $\chi_\Gamma^*(x) \equiv \int \chi_\Gamma(x) d\mu = \mu(\Gamma) > 0$ holds for a.e. point x . For the general invariant case, we prove by contradiction. Assume that the set $\Gamma_1 := \{x \in \Gamma \mid \chi_\Gamma^*(x) = 0\}$ has μ positive measure. Consider $A := \Gamma_1$, then $\chi_{\Gamma_1}^*(x)$ exists for μ a.e. x and $\int \chi_{\Gamma_1}^*(x) d\mu = \int \chi_{\Gamma_1}(x) d\mu = \mu(\Gamma_1) > 0$. Thus there exists $x \in \bigcup_{n \in \mathbb{Z}} f^n(\Gamma_1)$ such that $\chi_{\Gamma_1}^*(x) > 0$, since by definition $\chi_{\Gamma_1}^*(x) \equiv 0$ for all $x \in (\bigcup_{n \in \mathbb{Z}} f^n(\Gamma_1))^c$. Recall that $\chi_{\Gamma_1}^*(x)$ is f -invariant and thus if we let $y \in \Gamma_1$ such that $f^m(y) = x$ for some integer m , then $\chi_{\Gamma_1}^*(y) = \chi_{\Gamma_1}^*(f^{-m}(x)) = \chi_{\Gamma_1}^*(x) > 0$. Note that $\Gamma_1 \subset \Gamma$ implies $\chi_{\Gamma_1}(\cdot) \leq \chi_\Gamma(\cdot)$ and thus $\chi_{\Gamma_1}^*(\cdot) \leq \chi_\Gamma^*(\cdot)$. So

$$\chi_\Gamma^*(y) \geq \chi_{\Gamma_1}^*(y) > 0.$$

This contradicts the choice of Γ_1 since $y \in \Gamma_1$. □

Remark 3.2. We emphasize the main used technique in Proposition 3.1 that for an increasing sequence of integers $\{N_i\}_{i \geq 1}$, one sufficient condition to get nonlacunary is that $\lim_{i \rightarrow +\infty} \frac{i}{N_i} > 0$. Thus this can also be used to characterize the nonlacunary of hyperbolic times[15] instead of using Borel-Cantelli lemma.

Remark 3.3. We point out another equivalent statement of Proposition 3.1. That is, x satisfies $\lim_{i \rightarrow +\infty} \frac{t_{i+1}(x)}{t_i(x)} = 1 \Leftrightarrow$ for any $\epsilon > 0$, there exists large integer $N(x)$ such that for all $n \geq N(x)$, there is $t \in [n, n+n\epsilon)$ such that $t = t_i(x)$ for some i . The proof of this relation is trivial but the technique of the later version is also useful and has been essentially used in the proof to establish an (in)equality between metric entropy of hyperbolic ergodic measure and the number of hyperbolic periodic points in [10, 4, 13]. One main technique used in [10, 4, 13] is that, the recurrence time varies so slowly (linearly) that if the cardinality of a sequence of sets(for example, separated sets) is growing exponentially, then there exists a new sequence composed of their subsets such that the cardinality still grows exponentially and in every subset the recurrence times are same for all points.

4 Proof of Theorem 1.2: NS

Set $\tilde{\Lambda}_k = \text{supp}(\mu|_{\Lambda_k})$ and $\tilde{\Lambda} = \cup_{k=1}^{\infty} \tilde{\Lambda}_k$. Clearly, $f^{\pm 1}(\tilde{\Lambda}_k) \subset \tilde{\Lambda}_{k+1}$, and the sub-bundles $E^s(x)$, $E^u(x)$ depend continuously on $x \in \tilde{\Lambda}_k$. Moreover, $\tilde{\Lambda}$ is f -invariant with μ -full measure. Let $\Delta_k \subseteq \tilde{\Lambda}_k$ be the set of all points of x satisfying that

(i) recurrence times of x are well defined for the set $\Gamma := \tilde{\Lambda}_k$ and

$$(ii) \quad \lim_{i,j \rightarrow +\infty} \frac{t_{i+1}(x) - t_{-j-1}(x)}{t_i(x) - t_{-j}(x)} = 1. \quad (4.1)$$

By Poincaré Recurrence Theorem and Proposition 3.1, $\mu(\Delta_k) = \mu(\tilde{\Lambda}_k)$ and thus $\cup_{k \geq 1} \Delta_k$ is also a subset of M with full measure. So we only need to prove that for every fixed Δ_k with positive measure, all points in Δ_k satisfy the conditions of non-uniform specification property. Take and fix a point $x \in \Delta_k$.

Recall ε to be the number that appeared in the definition of Pesin set. Let $0 < \eta \leq \varepsilon/2$ and q be an η -slowing varying positive function, $\theta > 0$ and let m, n be two positive integers. Let $\tau = \frac{\theta q^{-2}(x)}{\varepsilon_0} > 0$. By Lemma 2.1 for this τ there exists a sequence $(\delta_k)_{k=1}^{+\infty}$ such that for any $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit there exists a unique τ -shadowing point.

Take and fix for $\tilde{\Lambda}_k$ a finite cover $\alpha_k = \{V_1, V_2, \dots, V_{r_k}\}$ by nonempty open balls V_i in M such that $\text{diam}(U_i) < \delta_{k+1}$ and $\mu(U_i) > 0$ where $U_i = V_i \cap \tilde{\Lambda}_k$, $i = 1, 2, \dots, r_k$. This is obtained from the definition of $\tilde{\Lambda}_k$. By Birkhoff ergodic theorem and the ergodicity of μ we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{h=0}^{n-1} \mu(f^{-h}(U_i) \cap U_j) = \mu(U_i)\mu(U_j) > 0. \quad (4.2)$$

Then one can take

$$X_{i,j} := \min\{h \in \mathbb{N} \mid h \geq 1, \mu(f^{-h}(U_i) \cap U_j) > 0\}.$$

By (4.2), $1 \leq X_{i,j} < +\infty$. Let

$$M_k = \max_{1 \leq i,j \leq r_k} X_{i,j}.$$

Note that M_k is a positive integer dependent on k, θ and the η -slowing varying positive function q , but independent of m, n . So

$$\frac{M_k}{m}, \frac{M_k}{n} \rightarrow 0 \quad (4.3)$$

as $m, n \rightarrow \infty$ respectively.

Now we start to find the needed shadowing periodic orbit. Take positive integers l_1 and l_2 such that

$$t_{-l_1} < -m \leq t_{-l_1+1} \quad \text{and} \quad t_{l_2} > n \geq t_{l_2-1}, \quad (4.4)$$

and take positive integers s_1 and s_2 such that

$$t_{-l_1-s_1} \leq (1 + \frac{2\eta}{\varepsilon})t_{-l_1} < t_{-l_1-s_1+1} \quad \text{and} \quad t_{l_2+s_2} \geq (1 + \frac{2\eta}{\varepsilon})t_{l_2} > t_{l_2+s_2-1}. \quad (4.5)$$

Since $f^{t_{-l_1-s_1}}(x), f^{t_{l_2+s_2}}(x) \in \tilde{\Lambda}_k$, we can take U_i and U_j such that

$$f^{t_{-l_1-s_1}}(x) \in U_i, f^{t_{l_2+s_2}}(x) \in U_j.$$

By (4.2), there exist $y \in U_j$ and $0 \leq N \leq M_k$ such that $f^N(y) \in U_i$.

Recall the property of Pesin blocks that $f^\pm(\Lambda_k) \subseteq \Lambda_{k+1}$. Thus, if $u \in \Lambda_k$, then $f^i(u) \in \Lambda_{k+|i|}$, $\forall i \in \mathbb{Z}$. Note that

$$f^{t_{-l_1-s_1}}(x), x, f^{t_{l_2+s_2}}(x), y, f^N(y) \in \tilde{\Lambda}_k \subseteq \Lambda_k$$

and

$$d(f^{t_{l_2+s_2}}(x), y) < \delta_{k+1}, \quad d(f^{t_{-l_1-s_1}}(x), f^N(y)) < \delta_{k+1}.$$

So

$$\begin{aligned} & f^{t_{-l_1-s_1}}(x) \in \Lambda_k, f^{t_{-l_1-s_1}+1}(x) \in \Lambda_{k+1}, \dots, f^{t_{-l_1-s_1}+i}(x) \in \Lambda_{\min\{k+i, k+l_1+s_1-i\}}, \\ & \dots, f^{-1}(x) \in \Lambda_{k+1}, x \in \Lambda_k, f(x) \in \Lambda_{k+1}, \dots, f^i(x) \in \Lambda_{\min\{k+i, k+l_2+s_2-i\}}, \\ & \dots, f^{t_{l_2+s_2}-1}(x) \in \Lambda_{k+1}, y \in \Lambda_k, \dots, f^{i-1}(y) \in \Lambda_{\min\{k+i, k+N-i\}}, \dots, f^{N-1}(y) \in \Lambda_{k+1}. \end{aligned}$$

Repeat the above sequence of points infinitely many times and thus we get a $(\delta_k)_{k=1}^{+\infty}$ pseudo-orbit. Then there exists a unique τ -shadowing point z . Note that $f^{t_{l_2+s_2}-t_{-l_1-s_1}+N}(z)$ is also a τ -shadowing point. So

$$f^p(z) = z$$

where $p = t_{l_2+s_2} - t_{-l_1-s_1} + N$.

Now we start to verify the conditions of non-uniform specification property for the above chosen shadowing point z . Define $K := K(\eta, \theta, x, m, n) = t_{l_2+s_2} - t_{-l_1-s_1} + M_k - m - n$. Clearly we have $p \leq m + n + K$ since $N \leq M_k$, and thus for the first condition of NS we only need to show

$$z \in \tilde{B}_m^n(x, \theta) := \cap_{i=-m}^n f^{-i} B(f^i(x), \theta q(f^i(x))^{-2}).$$

Firstly, we consider $0 \leq i \leq t_{l_2} - 1$ and calculate $d(f^i(x), f^i(z))$. More precisely, τ -shadowing implies that

$$\begin{aligned}
& d(f^i(x), f^i(z)) \\
& \leq \max\{\tau\varepsilon_{k+i}, \tau\varepsilon_{k+t_{l_2}+s_2-i}\} \\
& \leq \max\{\tau\varepsilon_i, \tau\varepsilon_{t_{l_2}+s_2-t_{l_2}}\} \quad (\text{note that } i \leq t_{l_2} \text{ and } \varepsilon_k \text{ is a decreasing sequence}) \\
& = \max\{\tau\varepsilon_0 e^{-i\varepsilon}, \tau\varepsilon_0 e^{-(t_{l_2}+s_2-t_{l_2})\varepsilon}\} \\
& \leq \max\{\tau\varepsilon_0 e^{-i\varepsilon}, \tau\varepsilon_0 e^{-2t_{l_2}\eta}\} \quad (\text{using (4.5)}) \\
& \leq \max\{\tau\varepsilon_0 e^{-2i\eta}, \tau\varepsilon_0 e^{-2i\eta}\} \quad (\text{using } 2\eta < \varepsilon \text{ and } i \leq t_{l_2}) \\
& = \tau\varepsilon_0 e^{-2i\eta} \\
& = \theta q^{-2}(x) e^{-2i\eta} \quad (\text{by the choice of } \tau) \\
& \leq \theta q(f^i(x))^{-2} \quad (\text{using } q(f^i(x)) \leq q(x) e^{i\eta}).
\end{aligned}$$

Secondly we can follow the similar method to show that for $t_{-l_1} + 1 \leq i \leq 0$,

$$d(f^i(x), f^i(z)) \leq \theta q(f^i(x))^{-2}.$$

Notice that $t_{-l_1} < -m$ and $n < t_{l_2}$ and thus $z \in \tilde{B}_m^n(x, \theta)$.

At the end we prove the second condition of NS:

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} = 0.$$

In fact, by using (4.1), (4.3), (4.4) and (4.5), we have

$$\begin{aligned}
& \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} \\
& \leq \limsup_{n \rightarrow +\infty} \frac{t_{l_2}+s_2 - t_{-l_1}-s_1}{m + n} + \limsup_{m, n \rightarrow +\infty} \frac{M_k}{m + n} - 1 \\
& \leq \limsup_{m, n \rightarrow +\infty} \frac{t_{l_2}+s_2 - t_{-l_1}-s_1}{t_{l_2}-1 - t_{-l_1}+1} + 0 - 1 \quad (\text{using (4.4), (4.3)}) \\
& = \limsup_{m, n \rightarrow +\infty} \left(\frac{t_{l_2}+s_2 - t_{-l_1}-s_1}{t_{l_2}+s_2-1 - t_{-l_1}-s_1+1} \cdot \frac{t_{l_2}+s_2-1 - t_{-l_1}-s_1+1}{t_{l_2} - t_{-l_1}} \cdot \frac{t_{l_2} - t_{-l_1}}{t_{l_2}-1 - t_{-l_1}+1} \right) - 1 \\
& = \limsup_{m, n \rightarrow +\infty} \frac{t_{l_2}+s_2-1 - t_{-l_1}-s_1+1}{t_{l_2} - t_{-l_1}} - 1 \quad (\text{using (4.1)}) \\
& \leq 1 + \frac{2\eta}{\varepsilon} - 1 = \frac{2\eta}{\varepsilon} \quad (\text{using (4.5)}).
\end{aligned}$$

Letting $\eta \rightarrow 0$, one has

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} = 0.$$

So we complete the proof. □

Remark 4.1. In particular, if μ is a mixing hyperbolic measure, we can replace inequality (4.2) by

$$\lim_{n \rightarrow +\infty} \mu(f^{-n}(U_i) \cap U_j) = \mu(U_i)\mu(U_j) > 0. \quad (4.6)$$

Then by (4.6) we can take a finite integer

$$X_{i,j} = \max\{n \in \mathbb{N} \mid n \geq 1, \mu(f^{-n}(U_i) \cap U_j) = 0\} + 1.$$

Let

$$M_k = \max_{1 \leq i,j \leq r_k} X_{i,j}.$$

Then for any $N \geq M_k$ there exist $y \in U_j$ such that $f^N(y) \in U_i$. So we can follow the above proof and then the non-uniform specification can be stronger: for any $p \geq n + m + K$, the non-uniform dynamical ball $\tilde{B}_m^n(x, \theta)$ contains a periodic point with period p .

5 Proof of Theorem 1.9: GNS

In this section we prove Theorem 1.9. Before, we show two propositions as follows.

Proposition 5.1. *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism. Then for any small $0 < \sigma < 1$, there is a subset Λ_σ^* with $\mu(\Lambda_\sigma^*) > 1 - \sigma$ such that for every $x \in \Lambda_\sigma^*$, any small $\eta > 0$, any $\theta_* > 0$ and any integer m, n , there exists $K := K(\eta, \theta_*, x, m, n)$ satisfying*

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta_*, x, m, n)}{m + n} = 0$$

and so that the following holds: given points x_1, x_2, \dots, x_k in Λ_σ^* and positive integers $m_1, \dots, m_k, n_1, \dots, n_k$, there are numbers $p_{*i} \geq 0$ ($1 \leq i \leq k$) with

$$\sum_{i=1}^k p_{*i} \leq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$$

(in particular if μ is mixing, for any $t \geq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$ there is p_{*i} with $\sum_{i=1}^k p_{*i} = t$) and there exists a periodic point z_* with period $p_* = \sum_{j=1}^k (n_j + m_j + p_{*j})$ such that

$$z_* \in \cap_{j=-m_1}^{n_1} f^{-j}(B(f^j(x_1), \theta_* e^{-2|j|\eta}))$$

and for $2 \leq i \leq k$,

$$f^{\sum_{j=1}^{i-1} (n_j + p_{*j}) + \sum_{j=2}^i m_j}(z_*) \in \cap_{j=-m_i}^{n_i} f^{-j}B(f^j(x_i), \theta_* e^{-2|j|\eta}).$$

This property is also valid for one side case.

Proof The proof is a generalization of that of Theorem 1.2. Recall ε to be the number that appeared in the definition of Pesin set. Recall Δ_{k_*} to be the set introduced in the proof of Theorem 1.2 and note that if k_* is large then the measure of Δ_{k_*} is close

to 1. Let k_* be large enough such that Δ_{k_*} satisfies $\mu(\Delta_{k_*}) > 1 - \sigma$. We will prove this Δ_{k_*} is the required Λ_σ^* .

Let $\tau = \frac{\theta_*}{\varepsilon_0} > 0$. By Lemma 2.1 there exists a sequence $(\delta_k)_{k=1}^{+\infty}$ such that for any $(\cdot)_{k=1}^{+\infty}$ pseudo-orbit there exists a unique τ -shadowing point.

Let $x \in \Delta_{k_*}$, $0 < \eta \leq \varepsilon/2$ and m, n be two positive integers. Note that δ_{k_*} dependent on $\tau = \frac{\theta_*}{\varepsilon_0}$ and Λ_{k_*} , but independent of x . This is different to the one in the proof of Theorem 1.2. Let $K(\eta, \theta_*, x, m, n)$ be the number defined as in the proof of Theorem 1.2. (Note that the choices of l_1, s_1 and l_2, s_2 only depends on Δ_{k_*}, x, m, n , and the choice of M_{k_*} only depends on δ_{k_*+1} and thus $K(\eta, \theta_*, x, m, n)$ only depends Δ_{k_*} and $\tau = \frac{\theta_*}{\varepsilon_0}$). So this $K(\eta, \theta_*, x, m, n)$ also satisfies

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta_*, x, m, n)}{m + n} = 0.$$

Re-denote the l_1, s_1 and l_2, s_2 with respect to x in the proof of Theorem 1.2 by $l_1(x), s_1(x)$ and $l_2(x), s_2(x)$.

Given points x_1, x_2, \dots, x_k in Δ_{k_*} and positive integers $m_1, \dots, m_k, n_1, \dots, n_k$, similar as the proof of Theorem 1.2, we can take

$$y_i, f^{N_i}(y_i) \in \tilde{\Lambda}_{k_*}$$

with $0 \leq N_i \leq M_{k_*}$ such that

$$d(f^{t_{l_2(x_i)+s_2(x_i)}}(x_i), y_i) < \delta_{k_*+1}, \quad d(f^{t_{-l_1(x_{i+1})-s_1(x_{i+1})}}(x_{i+1}), f^{N_i}(y_i)) < \delta_{k_*+1}, \quad \forall 1 \leq i \leq k_*,$$

where $x_{k+1} = x_1$. Note that

$$f^{t_{-l_1}-s_1}(x_i), x, f^{t_{l_2}+s_2}(x_i), y_i, f^{N_i}(y_i) \in \tilde{\Lambda}_{k_*} \subseteq \Lambda_{k_*}.$$

Similar as the proof of Theorem 1.2, by shadowing lemma there is a periodic point z_* with period $p_* = \sum_{j=1}^k (t_{l_2(x_j)+s_2(x_j)} - t_{l_1(x_j)+s_1(x_j)} + N_j)$ such that

$$z_* \in \cap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \tau \varepsilon_0 e^{-2|j|\eta}) = \cap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta_* e^{-2|j|\eta})$$

and for $2 \leq i \leq k$,

$$f^{\sum_{j=1}^{i-1} (t_{l_2(x_j)+s_2(x_j)} + N_j) - \sum_{j=2}^i t_{l_1(x_j)+s_1(x_j)}}(z_*) \in \cap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta_* e^{-2|j|\eta}).$$

Let $p_{*i} = t_{l_2(x_i)+s_2(x_i)} - n_i + N_i - t_{-l_1(x_{i+1})-s_1(x_{i+1})} - m_{i+1}$ where $m_{k+1} = m_1$ and thus $\sum_{i=1}^k p_{*i} = \sum_{i=1}^k (t_{l_2(x_i)+s_2(x_i)} - t_{-l_1(x_i)-s_1(x_i)} + N_i - m_i - n_i) \leq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$. Then the periodic point z_* satisfies that its period is $p_* = \sum_{j=1}^k (n_j + m_j + p_{*j})$,

$$z_* \in \cap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta_* e^{-2|j|\eta})$$

and for $2 \leq i \leq k$,

$$\begin{aligned} f^{\sum_{j=1}^{i-1} (n_j + p_{*j}) + \sum_{j=2}^i m_j}(z_*) &= f^{\sum_{j=1}^{i-1} (t_{l_2(x_j)+s_2(x_j)} + N_j) - \sum_{j=2}^i t_{l_1(x_j)+s_1(x_j)}}(z_*) \\ &\in \cap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta_* e^{-2|j|\eta}). \quad \square \end{aligned}$$

Proposition 5.2. *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism. Then for any small $0 < \sigma < 1$, any small $\eta > 0$, any η -slowing varying positive function q (that is, $q(f^{\pm 1}(x)) \leq e^\eta q(x)$) and any $\theta > 0$, there is a subset Λ_σ with $\mu(\Lambda_\sigma) > 1 - \sigma$ such that for every $x \in \Lambda_\sigma$, any integer m, n , there exists $K_* := K_*(\eta, \theta, x, m, n)$ satisfying*

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K_*(\eta, \theta, x, m, n)}{m + n} = 0$$

and so that the following holds: given points x_1, x_2, \dots, x_k in Λ_σ and positive integers $m_1, \dots, m_k, n_1, \dots, n_k$, there is $p_{*i} \geq 0$ with

$$\sum_{i=1}^k p_{*i} \leq \sum_{i=1}^k K_*(\eta, \theta, x_i, m_i, n_i)$$

(in particular if μ is mixing, for any $t \geq \sum_{i=1}^k K_*(\eta, \theta, x_i, m_i, n_i)$ there is p_{*i} with $\sum_{i=1}^k p_{*i} = t$) and a periodic point z_* with period $p_* = \sum_{j=1}^k (n_j + m_j + p_{*j})$ such that

$$z_* \in \tilde{B}_{m_1}^{n_1}(x_1, \theta) := \cap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta q(f^j(x_1))^{-2})$$

and for $2 \leq i \leq k$,

$$f^{\sum_{j=1}^{i-1} (n_j + p_{*j}) + \sum_{j=2}^i m_j}(z_*) \in \tilde{B}_{m_i}^{n_i}(x_i, \theta) := \cap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta q(f^j(x_i))^{-2}).$$

This property is also valid for one side case.

Proof Recall ε to be the number that appeared in the definition of Pesin set. Let $0 < \eta \leq \varepsilon/2$ and q be an η -slowing varying positive function. Recall $\Lambda_{\frac{\sigma}{2}}^*$ to be the set introduced in the proof of Proposition 5.1 for $\frac{\sigma}{2}$. Let $\theta_* > 0$ be small enough such that we can take $\Lambda_\sigma \subseteq \Lambda_{\frac{\sigma}{2}}^*$ with $\mu(\Lambda_\sigma) > 1 - \sigma$ and every point $x \in \Lambda_\sigma$ satisfies $\theta q^{-2}(x) \geq \theta_*$.

Let $x \in \Lambda_\sigma \subseteq \Lambda_{\frac{\sigma}{2}}^*$, m, n be two positive integers. Let $K(\eta, \theta_*, x, m, n)$ be the number as in Proposition 5.1, then if we can take $K_*(\eta, \theta, x, m, n) := K(\eta, \theta_*, x, m, n)$ and thus this $K_*(\eta, \theta, x, m, n)$ also satisfies

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K_*(\eta, \theta, x, m, n)}{m + n} = 0.$$

Re-denote the l_1, s_1 and l_2, s_2 with respect to x in the proof of Theorem 1.2 by $l_1(x), s_1(x)$ and $l_2(x), s_2(x)$.

Given points x_1, x_2, \dots, x_k in $\Lambda_\sigma \subseteq \Lambda_{\frac{\sigma}{2}}^*$ and positive integers $m_1, \dots, m_k, n_1, \dots, n_k$, by Proposition 5.1 there is $p_{*i} \geq 0$ with

$$\sum_{i=1}^k p_{*i} \leq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$$

(in particular if μ is mixing, for any $t \geq \sum_{i=1}^k K(\eta, \theta_*, x_i, m_i, n_i)$ there is p_{*i} with $\sum_{i=1}^k p_{*i} = t$) and there is a periodic point z_* with period $p_* = \sum_{j=1}^k (n_j + m_j + p_{*j})$,

$$z_* \in \cap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta_* e^{-2|j|\eta})$$

and for $2 \leq i \leq k$,

$$f^{\sum_{j=1}^{i-1}(n_j+p_{*j})+\sum_{j=2}^i m_j}(z_*) \in \cap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta_* e^{-2|j|\eta}).$$

Since $x_i \in \Lambda_\sigma$, then $\theta q^{-2}(x_i) \geq \theta_*$. Using $q(f^j(x_i)) \leq e^{|j|\eta} q(x_i)$, we have

$$\begin{aligned} z_* &\in \cap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta_* e^{-2|j|\eta}) \subseteq \cap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta q^{-2}(x_1) e^{-2|j|\eta}) \\ &\subseteq \cap_{j=-m_1}^{n_1} f^{-j} B(f^j(x_1), \theta q^{-2}(f^j(x_1))) \end{aligned}$$

and similarly for $2 \leq i \leq k$,

$$f^{\sum_{j=1}^{i-1}(n_j+p_{*j})+\sum_{j=2}^i m_j}(z_*) \in \cap_{j=-m_i}^{n_i} f^{-j} B(f^j(x_i), \theta q^{-2}(f^j(x_i))).$$

Now we start to prove Theorem 1.9.

Proof of Theorem 1.9 We use Proposition 5.2 to give a proof. Let Λ_σ be a *fixed* positive μ -measure set from Proposition 5.2. Note that $\cup_{j \geq 0} f^j(\Lambda_\sigma)$ is a full μ -measure set by the ergodicity of μ . We consider $x \in \cup_{j \geq 0} f^j(\Lambda_\sigma)$. Clearly we can take a finite (and fixed) number $t(x) \geq 0$ (which can be chosen the first time) such that $x \in f^{t(x)}(\Lambda_\sigma)$ and thus $f^{-t(x)}(x) \in \Lambda_\sigma$. Let $K_*(\eta, \theta, f^{-t(x)}(x), m, n + t(x))$ be the number as in Proposition 5.2 and define

$$K(\eta, \theta, x, m, n) := K_*(\eta, \theta, f^{-t(x)}(x), m, n + t(x)) + t(x).$$

Then $K(\eta, \theta, x, m, n)$ satisfies

$$\begin{aligned} &\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m + n} \\ &= \lim_{\eta \rightarrow 0} \limsup_{m, n+t(x) \rightarrow +\infty} \frac{K_*(\eta, \theta, f^{-t(x)}(x), m, n + t(x)) + t(x)}{m + n + t(x)} = 0. \end{aligned}$$

Given $x_1, \dots, x_k \in \cup_{j \geq 0} f^j(\Lambda_\sigma)$ and positive integers $m_1, m_2, \dots, m_k, n_1, \dots, n_k$ large enough, we consider points $f^{-t(x_1)}(x_1), \dots, f^{-t(x_k)}(x_k) \in \Lambda_\sigma$ and positive integers

$$m_1, m_2, \dots, m_k, n_1 + t(x_1), \dots, n_k + t(x_k),$$

by Proposition 5.2 there is $p_{*i} \geq 0$ with

$$\sum_{i=1}^k p_{*i} \leq \sum_{i=1}^k K_*(\eta, \theta, f^{-t(x_i)}(x_i), m_i, n_i + t(x_i))$$

(in particular if μ is mixing, for any $t \geq \sum_{i=1}^k K_*(\eta, \theta, f^{-t(x_i)}(x_i), m_i, n_i + t(x_i))$ there is p_{*i} with $\sum_{i=1}^k p_{*i} = t$) and a periodic point z_* with period

$$p_* = \sum_{j=1}^k (m_j + n_j + t(x_j) + p_{*j})$$

such that

$$z_* \in \tilde{B}_{m_1}^{n_1+t(x_1)}(f^{-t(x_1)}(x_1), \theta) := \cap_{j=-m_1}^{n_1+t(x_1)} f^{-j} B(f^{j-t(x_1)}(x_1), \theta q(f^{j-t(x_1)}(x_1))^{-2})$$

and for $2 \leq i \leq k$,

$$\begin{aligned} f^{\sum_{j=1}^{i-1} (n_j+t(x_j)+p_{*j})+\sum_{j=2}^i m_j}(z_*) &\in \tilde{B}_{m_i}^{n_i+t(x_i)}(f^{-t(x_i)}(x_i), \theta) \\ &= \cap_{j=-m_i}^{n_i+t(x_i)} f^{-j} B(f^{j-t(x_i)}(x_i), \theta q(f^{j-t(x_i)}(x_i))^{-2}). \end{aligned}$$

Let $p_i = p_{*i} + t(x_{i+1})$, $p = p_*$ where $x_{k+1} = x_1$, then

$$\begin{aligned} \sum_{i=1}^k p_i &\leq \sum_{i=1}^k K_*(\eta, \theta, f^{-t(x_i)}(x_i), m_i, n_i + t(x_i)) + \sum_{i=1}^k t(x_{i+1}) \\ &= \sum_{i=1}^k K(\eta, \theta, x_i, m_i, n_i + t(x_i)). \end{aligned}$$

Let $z = f^{t(x_1)}(z_*)$, then z is the needed periodic point. More precisely,

$$\begin{aligned} z &= f^{t(x_1)}(z_*) \in f^{t(x_1)} \tilde{B}_{m_1}^{n_1+t(x_1)}(f^{-t(x_1)}(x_1), \theta) \\ &= f^{t(x_1)} \cap_{j=-m_1}^{n_1+t(x_1)} f^{-j} B(f^{j-t(x_1)}(x_1), \theta q(f^{j-t(x_1)}(x_1))^{-2}) \\ &= \cap_{j=-m_1-t(x_1)}^{n_1} f^{-j} B(f^j(x_1), \theta q(f^j(x_1))^{-2}) \subseteq \tilde{B}_{m_1}^{n_1}(x_1, \theta) \end{aligned}$$

and similarly for $2 \leq i \leq k$, we have

$$\begin{aligned} f^{\sum_{j=1}^{i-1} (n_j+p_j)+\sum_{j=2}^i m_j}(z) &= f^{\sum_{j=1}^{i-1} (n_j+p_j)+\sum_{j=2}^i m_j+t(x_1)}(z_*) \\ &= f^{t(x_i)} \circ f^{\sum_{j=1}^{i-1} (n_j+t(x_j)+p_{*j})+\sum_{j=2}^i m_j}(z_*) \\ &\in f^{t(x_i)} \tilde{B}_{m_i}^{n_i+t(x_i)}(f^{-t(x_i)}(x_i), \theta) \\ &= f^{t(x_i)} \cap_{j=-m_i}^{n_i+t(x_i)} f^{-j} B(f^{j-t(x_i)}(x_i), \theta q(f^{j-t(x_i)}(x_i))^{-2}) \\ &= \cap_{j=-m_i-t(x_i)}^{n_i} f^{-j} B(f^j(x_i), \theta q(f^j(x_i))^{-2}) \subseteq \tilde{B}_{m_i}^{n_i}(x_i, \theta). \quad \square \end{aligned}$$

Remark 5.3. From the above discussion, in fact one also has a precise description of p_{*i} :

$$p_{*i} \leq K_*(\eta, \theta, x_i, m, n) + K_*(\eta, \theta, x_{i+1}, m, n)$$

and then

$$p_i \leq K(\eta, \theta, x_i, m, n) + K(\eta, \theta, x_{i+1}, m, n).$$

In other words, p_{*i}, p_i only depends on the point x_i and next point x_{i+1} .

6 Proof of Theorem 1.11

To prove Theorem 1.11 we need the exponentially shadowing lemma in [24] (C^1 Pesin theory). Before that we introduce some notions. Given $x \in M$ and $n \in \mathbb{N}$, let

$$\{x, n\} := \{f^j(x) \mid j = 0, 1, \dots, n\}.$$

In other words, $\{x, n\}$ represents the orbit segment from x to $f^n(x)$ with length n . For a sequence of points $\{x_i\}_{i=-\infty}^{+\infty}$ in M and a sequence of positive integers $\{n_i\}_{i=-\infty}^{+\infty}$, we call $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ a δ -pseudo-orbit, if $d(f^{n_i}(x_i), x_{i+1}) < \delta$ for all i . Given $\varepsilon > 0$ and $\tau > 0$, we call a point $x \in M$ an (exponentially) (τ, ε) -shadowing point for a pseudo-orbit $\{x_i, n_i\}_{i=-\infty}^{+\infty}$, if

$$d(f^{c_i+j}(x), f^j(x_i)) < \tau \cdot e^{-\min\{j, n_i-j\}\varepsilon},$$

$\forall j = 0, 1, 2, \dots, n_i$ and $\forall i \in \mathbb{Z}$, where c_i is defined as

$$c_i = \begin{cases} 0, & \text{for } i = 0 \\ \sum_{j=0}^{i-1} n_j, & \text{for } i > 0 \\ -\sum_{j=i}^{-1} n_j, & \text{for } i < 0. \end{cases} \quad (6.7)$$

Lemma 6.1. *Let us assume same conditions as in Theorem 1.11. Then for each $\sigma > 0$, there exist a compact set $\Lambda_\sigma \subseteq M$, $\varepsilon_\sigma > 0$ and $T_\sigma \in \mathbb{N}$ such that $\mu(\Lambda_\sigma) > 1 - \sigma$ and following (Exponentially) Shadowing Lemma holds. For $\forall \tau > 0$, there exists $\delta = \delta(\sigma, \tau) > 0$ such that if a δ -pseudo-orbit $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ satisfies $n_i \geq T_\sigma$ and $x_i, f^{n_i}(x_i) \in \Lambda_\sigma$ for all i , then there exists an (exponentially) $(\tau, \varepsilon_\sigma)$ -shadowing point $x \in M$ for $\{x_i, n_i\}_{i=-\infty}^{+\infty}$. If further $\{x_i, n_i\}_{i=-\infty}^{+\infty}$ is periodic, i.e., there exists an integer $m > 0$ such that $x_{i+m} = x_i$ and $n_{i+m} = n_i$ for all i , then the shadowing point x can be chosen to be periodic.*

Proof of Theorem 1.11 Here we point out that the result of Lemma 6.1 is weaker than the statements of Katok's shadowing lemma since it holds only for the pseudo-orbit whose beginning and ending points in the same Pesin block. However, Lemma 6.1 is enough to prove Theorem 1.11, since it also can deduce all propositions in Section 5. In other words, every ergodic measure of a homeomorphism with the property stated as in Lemma 6.1 has (generalized) non-uniform specification. Here we only give a proof of non-uniform specification (Definition 1.1).

Since the given hyperbolic measure μ is ergodic, the number ε_σ in Lemma 6.1 can be chosen independent on σ from Remark 1.4 in [24] and thus we can take a fixed number ε . In other word, for each $\sigma > 0$, there exist a compact set $\Lambda_\sigma \subseteq M$ and $T_\sigma \in \mathbb{N}$ such that $\mu(\Lambda_\sigma) > 1 - \sigma$ and (Exponentially) Shadowing Lemma holds as in Lemma 6.1 for $\varepsilon_\sigma \equiv \varepsilon$.

Set $\tilde{\Lambda}_\sigma = \text{supp}(\mu|_{\Lambda_\sigma})$ and $\tilde{\Lambda} = \cup_{\sigma>0} \tilde{\Lambda}_\sigma$. Clearly, $\tilde{\Lambda}$ is of μ -full measure. Let $\Delta_\sigma \subseteq \tilde{\Lambda}_\sigma$ be the set of all points whose recurrence times are well defined for $\Gamma = \tilde{\Lambda}_\sigma$ and satisfy

$$\lim_{i,j \rightarrow +\infty} \frac{t_{i+1}(x) - t_{-j-1}(x)}{t_i(x) - t_{-j}(x)} = 1. \quad (6.8)$$

By Poincaré Recurrence Theorem and Proposition 3.1, $\mu(\Delta_\sigma) = \mu(\tilde{\Lambda}_\sigma)$ and thus $\cup_{\sigma>0} \Delta_\sigma$ is a set with full measure. So we only need to prove that for every fixed Δ_σ with positive measure, all points in Δ_σ satisfy the conditions of non-uniform specification property. Take and fix a point $x \in \Delta_\sigma$.

Let $0 < \eta \leq \varepsilon/2$ and q be an η -slowing varying positive function, $\theta > 0$ and let m, n be two positive integers. We may assume that $m, n \geq T_\sigma$ (Otherwise, consider $m' = m + T_\sigma$, $n' = n + T_\sigma$). Let $\tau = \theta q^{-2}(x) > 0$. Then for this τ there exists $\delta = \delta(\tau, \sigma) > 0$ satisfying Lemma 6.1.

Take and fix for $\tilde{\Lambda}_\sigma$ a finite cover $\alpha_\sigma = \{V_1, V_2, \dots, V_{r_\sigma}\}$ by nonempty open balls V_i in M such that $\text{diam}(U_i) < \delta$ and $\mu(U_i) > 0$ where $U_i = V_i \cap \tilde{\Lambda}_\sigma$, $i = 1, 2, \dots, r_\sigma$. Since μ is f -ergodic, by Birkhoff ergodic theorem we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{h=0}^{n-1} \mu(f^{-h}(U_i) \cap U_j) = \mu(U_i)\mu(U_j) > 0. \quad (6.9)$$

Then take

$$X_{i,j} = \min\{h \in \mathbb{N} \mid h \geq \mathbf{T}_\sigma, \mu(f^{-h}(U_i) \cap U_j) > 0\}.$$

By (6.9), $\mathbf{T}_\sigma \leq X_{i,j} < +\infty$. Let

$$M_\sigma = \max_{1 \leq i,j \leq r_\sigma} X_{i,j}.$$

Note that M_σ is dependent on σ, θ and the η -slowing varying positive function q , but independent of m, n . So

$$\frac{M_\sigma}{m}, \frac{M_\sigma}{n} \rightarrow 0 \quad (6.10)$$

as $m, n \rightarrow \infty$ respectively.

Take positive integers l_1 and l_2 such that

$$t_{-l_1} < -m \leq t_{-l_1+1} \quad \text{and} \quad t_{l_2} > n \geq t_{l_2-1}. \quad (6.11)$$

Take positive integers s_1 and s_2 such that

$$t_{-l_1-s_1} \leq (1 + \frac{2\eta}{\varepsilon})t_{-l_1} < t_{-l_1-s_1+1} \quad \text{and} \quad t_{l_2+s_2} \geq (1 + \frac{2\eta}{\varepsilon})t_{l_2} > t_{l_2+s_2-1}. \quad (6.12)$$

Take $K := K(\eta, \theta, x, m, n) = t_{l_2+s_2} - t_{-l_1-s_1} + M_\sigma - m - n$. The calculation of

$$\lim_{\eta \rightarrow 0} \limsup_{m, n \rightarrow +\infty} \frac{K(\eta, \theta, x, m, n)}{m+n} = 0$$

is similar as in the proof of Theorem 1.2 by using (6.11), (6.10), (6.8) and (6.12). We omit the details.

Since $f^{t_{-l_1-s_1}}(x), f^{t_{l_2+s_2}}(x) \in \tilde{\Lambda}_\sigma$, we can take U_i and U_j such that

$$f^{t_{-l_1-s_1}}(x) \in U_i, f^{t_{l_2+s_2}}(x) \in U_j.$$

By (6.9), there exist $y \in U_j$ and $\mathbf{T}_\sigma \leq N \leq M_\sigma$ such that $f^N(y) \in U_i$.

Note that

$$\begin{aligned} f^{t_{-l_1-s_1}}(x), x, f^{t_{l_2+s_2}}(x), y, f^N(y) &\in \tilde{\Lambda}_k \subseteq \Lambda_k, \\ -t_{-l_1-s_1} \geq m \geq T_\sigma, \quad t_{l_2+s_2} \geq n \geq T_\sigma, \quad N \geq T_\sigma \end{aligned}$$

and

$$d(f^{t_{l_2+s_2}}(x), y) < \delta, \quad d(f^{t_{-l_1-s_1}}(x), f^N(y)) < \delta.$$

So if we repeat the orbit segments of

$$\{f^{t_{-l_1-s_1}}(x), -t_{-l_1-s_1}\}, \quad \{x, t_{l_2+s_2}\}, \quad \{y, N\}$$

infinite times, then we get a periodic δ pseudo-orbit. Then by Lemma 6.1 there exists a periodic point z with period $p = t_{l_2+s_2} - t_{-l_1-s_1} + N$ such that

$$d(f^j(x), f^j(z)) < \tau \cdot e^{-\min\{j, t_{l_2+s_2}-j\}\varepsilon},$$

$\forall j = 0, 1, 2, \dots, t_{l_2+s_2}$ and

$$d(f^j(x), f^j(z)) < \tau \cdot e^{-\min\{-j, -t_{-l_1-s_1}+j\}\varepsilon},$$

$\forall j = t_{-l_1-s_1}, \dots, -2, -1, 0$.

Now we start to verify the conditions of non-uniform specification property. Clearly we have $p \leq m + n + K$ since $N \leq M_\sigma$, and thus we only need to show

$$z \in \tilde{B}_m^n(x, \theta) := \cap_{i=-m}^n f^{-i} B(f^i(x), \theta q(f^i(x))^{-2}).$$

Firstly, we consider $0 \leq i \leq t_{l_2} - 1$ and calculate $d(f^i(x), f^i(z))$. More precisely,

$$\begin{aligned} &d(f^i(x), f^i(z)) \\ &\leq \max\{\tau e^{-i\varepsilon}, \tau e^{-(t_{l_2+s_2}-i)\varepsilon}\} \\ &\leq \max\{\tau e^{-i\varepsilon}, \tau e^{-(t_{l_2+s_2}-t_{l_2})\varepsilon}\} \quad (\text{using } i \leq t_{l_2}) \\ &\leq \max\{\tau e^{-i\varepsilon}, \tau e^{-2t_{l_2}\eta}\} \quad (\text{using (6.12)}) \\ &\leq \max\{\tau e^{-2i\eta}, \tau e^{-2i\eta}\} \quad (\text{using } 2\eta < \varepsilon \text{ and } i \leq t_{l_2}) \\ &= \tau e^{-2i\eta} \\ &= \theta q^{-2}(x) e^{-2i\eta} \quad (\text{by the choice of } \tau) \\ &\leq \theta q(f^i(x))^{-2} \quad (\text{using } q(f^i(x)) \leq q(x) e^{i\eta}). \end{aligned}$$

Secondly we can follow the similar method to show that for $t_{-l_1} + 1 \leq i \leq 0$,

$$d(f^i(x), f^i(z)) \leq \theta q(f^i(x))^{-2}.$$

Notice that $t_{-l_1} < -m$ and $n < t_{l_2}$ and thus $z \in \tilde{B}_m^n(x, \theta)$. □

Remark 6.2. From the proofs of Theorem 1.11 and Theorem 1.2, we point out that for an invariant measure of a homeomorphism $f : X \rightarrow X$ on a compact metric space, a sufficient condition of non-uniform specification for μ is that: There exists $\varepsilon > 0$ such that for any $\sigma > 0$, there is a subset Γ_σ with μ positive measure larger than $1 - \sigma$ such that for any $\tau > 0$ and any integer m, n , if $f^{-m}(x), x, f^n(x) \in \Gamma_\sigma$ there exists $L := L(\sigma, \tau, x, m, n)$ such that:

(i) there exists a periodic point z with period $p \leq m + n + L$ such that

$$d(f^j(x), f^j(z)) < \tau \cdot e^{-\min\{j, n-j\}\varepsilon},$$

$\forall j = 0, 1, 2, \dots, n$ and

$$d(f^j(x), f^j(z)) < \tau \cdot e^{-\min\{-j, m+j\}\varepsilon},$$

$\forall j = -m, \dots, -2, -1, 0$;

(ii) the dependence of L on m, n satisfies

$$\limsup_{m, n \rightarrow +\infty} \frac{L(\sigma, \tau, x, m, n)}{m + n} = 0.$$

In particular, exponentially shadowing is such a sufficient condition. Note that from the proofs of Theorem 1.11 and Theorem 1.2, L is the number M_k or M_σ independent of m, n . We can also give a similar sufficient condition with several orbit segments for generalized non-uniform specification (we omit the details). There are also some other related papers [26, 6, 8, 4, 24] that exponential shadowing plays important roles. Exponential closing (which is the particular exponential shadowing for one pseudo orbit segment) has played crucial roles in [26, 6, 8] to prove that Lyapunov exponents of ergodic measures can be approximated by ones of periodic measures and it has also been used in [4, 24, 8] to calculate Hölder functions or Hölder cocycles to get some convergence properties for proving corresponding Livšic Theorem.

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